

# A fast scheme for the implementation of the quantum Rabi model with trapped ions

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**Abstract.** We show how to produce a fast quantum Rabi model with trapped ions. Its importance resides not only in the acceleration of the phenomena that may be achieved with these systems, from quantum gates to the generation of nonclassical states of the vibrational motion of the ion, but also in reducing unwanted effects such as the decay of coherences that may appear in such systems.

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## 1. Introduction

Trapped ions are considered one of the best candidates to perform quantum information processing. By interacting them with laser beams they are, somehow, easy to manipulate, which makes them an excellent choice for the production of nonclassical states of their vibrational motion.

The trapping of individual ions also offers many possibilities in spectroscopy [1], in the research of frequency standards [2, 3], in the study of quantum jumps [4], and in the generation of nonclassical vibrational states of the ion [5], to name some applications. To make the ions more stable in the trap, increasing the time of confinement, and also to avoid undesirable random motions, it is needed that the ion be in its vibrational ground state which may be accomplished by means of an adequate use of lasers.

Because of the high nonlinearities of the ion-laser interaction its theoretical treatment is a nontrivial problem [6–10]. Even in the simplest cases of interaction one has to employ physically motivated approximations in order to find a solution. A well-known example is the *Lamb-Dicke* approximation, in which the ion is considered to be confined within a region much smaller than the laser wavelength. Other examples are optical and vibrational rotating wave approximations that are usually performed in order to find simpler Hamiltonians.

Many treatments also assume a *weak coupling* approximation, such that, by tuning the laser frequency to integer multiples of the trap frequency results in effective (nonlinear) Hamiltonians of the Jaynes-Cummings type [11, 12], in which the centre-of-mass of the trapped ion plays the role of the field mode in cavity QED.

Recently it was shown that the quantum Rabi model could be engineered via the interaction of two laser beams with a trapped ion [13]. Pedernales *et al.* did it by slightly detuning both laser beams from the blue and red side bands, allowing them to construct a Hamiltonian of the Rabi type and reaching all the possible regimes. However, because the parameters involved are much smaller than the vibrational frequency of the ion,  $\nu$ , the ion can suffer losses that lead to the decay of Rabi oscillations [14, 15]. There have been attempts to explain such loss of coherences via laser intensity and phase fluctuations [16].

We will show here two approaches in which we can engineer a fast Quantum Rabi model (QRM), fast in the sense that the parameters involved in the interaction may be of the order of  $\nu$ . Instead of two off-resonant lasers [13], we use only one resonant beam.

## 2. Lamb-Dicke regime

We can write the Hamiltonian of the trapped ion as

$$H = H_{\text{vib}} + H_{\text{at}} + H_{\text{int}}, \quad (1)$$

where  $H_{\text{vib}}$  is the ion's center of mass vibrational energy,  $H_{\text{at}}$  is the ion internal energy, and  $H_{\text{int}}$  is the interaction energy between the ion and the laser. The vibrational motion can be approximated by a harmonic oscillator. Internally, the ion will be modelled by a

two level system. In the interaction between the ion and the laser beam, we will make the dipolar approximation, so we will write the interaction energy as  $-e\vec{r} \cdot \vec{E}$ , where  $-e\vec{r}$  is the dipolar momentum of the ion and  $\vec{E}$  is the electric field of the laser, that will be considered a plane wave. Thus, we write the Hamiltonian, after an optical rotating wave approximations as

$$H = \nu \hat{n} + \frac{\omega_0}{2} \sigma_z + \Omega [e^{i(kx - \omega_l t + \phi_l)} \sigma_+ + e^{-i(kx - \omega_l t + \phi_l)} \sigma_-]. \quad (2)$$

The first term in the Hamiltonian is the ion vibrational energy; in the ion vibrational energy, the operator  $\hat{n} = \hat{a}^\dagger \hat{a}$  is the number operator, and the ladder operators  $\hat{a}$  and  $\hat{a}^\dagger$  are given by the expressions

$$\hat{a} = \sqrt{\frac{\nu}{2}} \hat{x} + i \frac{\hat{p}}{\sqrt{2\nu}}, \quad \hat{a}^\dagger = \sqrt{\frac{\nu}{2}} \hat{x} - i \frac{\hat{p}}{\sqrt{2\nu}}, \quad (3)$$

where we have set the ion mass equal to one. Also, for simplicity, we have displaced the vibrational Hamiltonian by  $\nu/2$ , the vacuum energy, that without loss of generality may be disregarded.

The second term in the Hamiltonian corresponds to the ion internal energy; the matrices  $\sigma_z$ ,  $\sigma_+$ , and  $\sigma_-$  are the Pauli matrices, and obey the commutation relations

$$[\sigma_z, \sigma_\pm] = \pm 2\sigma_\pm, \quad [\sigma_+, \sigma_-] = \sigma_z, \quad (4)$$

and  $\omega_0$  is the transition frequency between the ground state and the excited state of the ion.

By considering the resonant condition,  $\omega_0 = \omega_l$ , and transforming to a picture rotating at  $\omega_l$  we obtain the Hamiltonian

$$H = \nu \hat{n} + \Omega [e^{i\phi_l} \hat{D}(i\eta) \sigma_+ + e^{-i\phi_l} \sigma_- \hat{D}^\dagger(i\eta)], \quad (5)$$

where we have defined the so-called Lamb-Dicke parameter

$$\eta = k \sqrt{\frac{1}{2m\nu}} \quad (6)$$

that is a measure of the amplitude of the oscillations of the ion with respect to the wavelength of the laser field represented by its wave vector  $k$ .

If we consider the condition  $\eta\sqrt{\bar{n}} \ll 1$ , where  $\bar{n}$  is the average number of vibrational quanta, we can expand the Glauber displacement operator [17] in Taylor series

$$\hat{D}(i\eta) \approx 1 + i\eta \hat{a}^\dagger + i\eta \hat{a}, \quad (7)$$

such that the Hamiltonian (2) reads

$$H \approx \nu \hat{n} + \Omega [e^{i\phi_l} \sigma_+ + e^{-i\phi_l} \sigma_-] + i\eta \Omega (\hat{a}^\dagger + \hat{a}) [e^{i\phi_l} \sigma_+ - e^{-i\phi_l} \sigma_-]. \quad (8)$$

By setting  $\phi_l = \pi$  and making now a rotation around the  $Y$  axis (by means of the transformation  $\exp(i\frac{\pi}{4}\sigma_y)$ ), with  $\sigma_y = i\sigma_- - \sigma_+$ , we obtain the usual form of the Rabi Hamiltonian

$$H = \nu \hat{n} - \Omega \sigma_z - i\eta \Omega (\hat{a}^\dagger + \hat{a})(\sigma_+ - \sigma_-) \quad (9)$$

If we take now  $\nu = -2\Omega$ , and we use the rotating wave approximation, the Hamiltonian reduces to the anti-Jaynes-Cummings (AJC) interaction Hamiltonian

$$H = -i\eta\Omega (\hat{a}\sigma_- - \sigma_+\hat{a}^\dagger). \quad (10)$$

On the other hand, if we set  $\phi_l = 0$  and follow the same procedure we obtain

$$H = \nu\hat{n} - \Omega\sigma_z - i\eta\Omega(\hat{a}^\dagger + \hat{a})(\sigma_+ - \sigma_-) \quad (11)$$

that, by taking  $\nu = 2\Omega$ , and using the rotating wave approximation now reduces to the Jaynes-Cummings (JC) interaction Hamiltonian

$$H = i\eta\Omega (\hat{a}\sigma_+ - \sigma_-\hat{a}^\dagger). \quad (12)$$

Up to here we have been able to construct the Rabi interaction, equation (8), with a set of parameters that do not allow all the regimes because  $\eta \ll 1$  only permits the JC and AJC interactions. However, we should stress that this is a much faster interaction than the one produced by Pedernales *et al.* [13] as  $\Omega$  is the order of  $\nu$ .

### 3. Fast Rabi Hamiltonian

We turn our attention again to the Hamiltonian given in equation (8) and set  $\phi_l = 0$

$$H = \nu\hat{n} + \Omega (\sigma_+\hat{D}(i\eta) + \sigma_-\hat{D}^\dagger(i\eta)), \quad (13)$$

we rewrite equation (13) in a notation where operators acting on the internal ionic levels are represented explicitly in terms of their matrix elements, as

$$H = \begin{pmatrix} \nu\hat{n} & \Omega\hat{D}(i\eta) \\ \Omega\hat{D}^\dagger(i\eta) & \nu\hat{n} \end{pmatrix} \quad (14)$$

and consider now the unitary operator [18, 19]

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{D}^\dagger(i\eta/2) & \hat{D}(i\eta/2) \\ -\hat{D}^\dagger(i\eta/2) & \hat{D}(i\eta/2) \end{pmatrix}. \quad (15)$$

It is possible to check after some algebra that

$$\mathcal{H}_{\text{QRM}} = THT^\dagger = \begin{pmatrix} \nu\hat{n} + \Omega + \frac{\nu\eta^2}{4} & \frac{\nu\eta}{2} (\hat{a} - \hat{a}^\dagger) + \frac{\delta}{2} \\ \frac{\nu\eta}{2} (\hat{a} - \hat{a}^\dagger) + \frac{\delta}{2} & \nu\hat{n} - \Omega + \frac{\nu\eta^2}{4} \end{pmatrix}, \quad (16)$$

that, after returning to matrix notation, reads

$$\mathcal{H}_{\text{QRM}} = \nu\hat{n} + \Omega\sigma_z + \frac{\nu\eta}{2} (\sigma_+ + \sigma_-) (\hat{a} - \hat{a}^\dagger) + \frac{\nu\eta^2}{4}, \quad (17)$$

that is nothing but the quantum Rabi Hamiltonian plus a constant term that can be disregarded. A solution for this model has been given recently by Braak [20]

It should be stressed now that in the above Hamiltonian we have not made any assumptions on the parameters  $\Omega$  and  $\eta$ .

The transformation (15) has already been used to find (families of) exact solutions to the QRM [21]. It has been also used to implement fast quantum gates in trapped ions [22].

This correspondence is very useful, since it enables one to map interesting properties of each model onto their counterparts in the other. For instance ways of realizing substantially faster logic gates for quantum information processing in a linear ion chain [22].

### 3.1. Effective Hamiltonian

Now we show how to produce a fast dispersive Hamiltonian. Pedernales *et al.* [13] showed that it is possible to build such a Hamiltonian by using two slightly off resonant laser beams tuned almost to the blue and red sidebands. However, as the parameters they used are in general much smaller than  $\nu$ , the dispersive interaction constant, may be very small. Here, we take advantage of the fact that the Hamiltonian given in (17) has not been approximated and therefore there are no restriction on the values of their parameters. By transforming the Hamiltonian (17) with the unitary operators [23]

$$\hat{U}_1 = e^{\epsilon_1(\hat{a}^\dagger\hat{\sigma}_+ - \hat{a}\hat{\sigma}_-)}, \quad \hat{U}_2 = e^{\epsilon_2(\hat{a}\hat{\sigma}_+ - \hat{a}^\dagger\hat{\sigma}_-)}; \quad (18)$$

with  $\epsilon_1, \epsilon_2 \ll 1$ ,

$$\hat{\mathcal{H}}_{\text{eff}} = \hat{U}_2 \hat{U}_1 \hat{\mathcal{H}}_{\text{QRM}} \hat{U}_1^\dagger \hat{U}_2^\dagger, \quad (19)$$

and setting

$$\epsilon_1 = \frac{\eta\nu}{2(\nu + 2\Omega)} \quad \epsilon_2 = \frac{\eta\nu}{2(2\Omega - \nu)}, \quad (20)$$

remaining up to first order in the expansion  $e^{\epsilon A} B e^{-\epsilon A} = B + \epsilon[A, B] + \frac{\epsilon^2}{2!}[A, [A, B]] + \dots \approx B + \epsilon[A, B]$ , *i.e.* doing a small rotation [23], we obtain the so-called dispersive Hamiltonian

$$\hat{\mathcal{H}}_{\text{eff}} = \nu \hat{a}^\dagger \hat{a} + \Omega \hat{\sigma}_z - \chi_{\text{QRM}} \hat{\sigma}_z (\hat{a}^\dagger \hat{a} + \frac{1}{2}) \quad (21)$$

where the effective interaction constant has the form

$$\chi_{\text{QRM}} = \frac{2\eta^2\nu^2\Omega}{4\Omega^2 - \nu^2}. \quad (22)$$

## 4. Conclusions

Note that most regimes may be achieved with this *fast* treatment: Jaynes-Cummings and anti Jaynes-Cummings were produced with the first method ( $\eta \ll 1$ ) and may also be produced with the last one, where the decoupling regime may also be achieved  $\nu \gg \eta\nu/2 \gg 2\Omega$ , the two-fold dispersive regime where  $\eta\nu/2 < \nu, 2\Omega, |2\Omega - \nu|, |2\Omega + \nu|$  may be considered, etc. It should again be stressed that, because decay actually happens in ion-laser interactions [14,16] it is of great importance to have fast interactions [22] in

order to minimize such unwanted interactions that avoid the generation of nonclassical states, quantum gates, and other important quantum effects.

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